MULTIPLE ALGEBRA.

PROFESSOR GIBBS'S masterly address upon the subject of 'multiple algebra' was too long and of too technical a nature for presentation in full to our readers, and the quotation of a few passages, and a brief summary of the bearings of the remainder, must suffice to acquaint them with its general drift and importance. His opening remarks were as follows: —

"It has been said that 'the human mind has never invented a labor-saving machine equal to algebra.' If this be true, it is but natural and proper that an age like our own, characterized by the multiplication of labor-saving machinery, should be distinguished by an unexampled development of this most refined and most beautiful of machines. That such has been the case, no one will question. The improvement has been in every part. Even to enumerate the principal lines of advance, would be a task for any one, — for me, an impossibility. But if we should ask in what direction the advance has been made, what is to characterize the development of algebra in our day, we may, I think, point to that broadening of its fields and methods which gives us 'multiple algebra.' Of the importance of this change in the conception of the office of algebra, it is hardly necessary to speak: that it is really characteristic of our time, will be most evident if we go back some two or three score years, to the time when the seeds were sown which are now yielding so abundant a harvest. The failure of Möbius, Hamilton, Grassmann, Saint-Venant, to make an immediate impression upon the course of mathematical thought in any way commensurate with the importance of their discoveries, is the most conspicuous evidence that the times were not ripe for the methods which they sought to introduce. A satisfactory theory of the imaginary quantities of ordinary algebra, which is essentially a simple case of multiple algebra, with difficulty obtained recognition in the first third of this century. We must observe that this 'double algebra,' as it has been called, was not sought for or invented, — it forced itself, unbidden, upon the attention of mathematicians, and with its rules already formed."

The speaker then gave a critical historical review of the different contributions of Hamilton, Möbius, Grassmann, Saint-Venant, Cauchy, Cayley, Hankel, the Peirces, father and son, and Sylvester, to these new methods of mathematical analysis, showing the additions and developments made by each to the various subjects.

In the second part of the paper, Professor Gibbs criticised the methods of some modern writers on these subjects, showing how they failed to grasp the full significance and bearings of the matters they were dealing with, being too much hampered by the old ideas and methods of simple algebra. We quote here a few sentences: —

"This fault has been denounced by Sylvester; and if any one thinks that I make too much of the stand-point from which we view the subject, I will refer him to the opening paragraphs of the lectures on 'universal algebra' in the sixth volume of the *American journal of mathematics*, where, with a wealth of illustration and an energy of diction which I cannot emulate, the most eloquent of mathematicians expresses his sense of the importance of the substitution of the idea of the matrix for that of the determinant. If this is so important, why was the idea of the matrix let slip? Of course, the writers on this subject had it to commence with. One cannot even define a determinant without the idea of a matrix. The simple fact is, that the writers on this subject have especially developed those ideas which are naturally expressed in simple algebra, and have postponed, or slurred over, or omitted altogether, those ideas which find their natural expression in multiple algebra. But in this subject, the latter happened to be the fundamental ideas, and those which ought to direct the whole course of thought." Many illustrations were then given of the applications of these ideas to cases in point.

The author introduced the third part of his paper as follows: "We have considered the subject a good while from the outside; we have glanced at the principal events in the history of multiple algebra; we have seen how the course of modern thought seems to demand its aid, how it is actually leaning toward it, and beginning to adopt its methods. It may be worth while to direct our attention more critically to multiple algebra itself, and to inquire into its essential character and its most important principles. I do not know that anything useful or interesting, which relates to multiple quantity, and can be symbolically expressed, falls outside the domain of multiple algebra. But if it is asked, what notions are to be regarded as fundamental? we must answer, here as elsewhere, those which are most simple and fruitful. Unquestionably, no relations are more so than those which are known by the names of addition and multiplication."

Then followed a long discussion of the fundamental conceptions and methods of modern math-

Abstract of an address before the section of mathematics and astronomy of the American association for the advancement of science at Buffalo, Aug. 19, 1886, by Prof. J. Willard Gibbs, of New Haven, Conn., vice-president of the section.

ematics, which nothing but publication in full could render intelligible, and that only to the mathematicians among our readers. To such, its full publication in the 'Proceedings' will prove of the greatest value.

The fourth part of the paper was devoted to consideration of some of the applications of multiple algebra. From this we quote the following: "First of all, geometry, and the geometrical sciences which treat of things having position in space, — kinematics, mechanics, astronomy, crystallography, — seem to demand a method of this kind, for position in space is essentially a multiple quantity, and can only be represented by simple quantities in an arbitrary and cumbersome manner. For this reason, and because our spatial intuitions are more developed than those of any other class of mathematical relations, these subjects are especially adapted to introduce the student to the methods of multiple algebra. Here nature herself takes us by the hand, and leads us along by easy steps, as a mother teaches her child to walk. In the contemplation of these subjects, Möbius, Hamilton, and Grassmann formed their algebras, although the philosophical mind of the last was not satisfied until he had produced a system unfettered by any spatial relations. It is probably in connection with these subjects that the notions of multiple algebra are most widely disseminated. Maxwell's 'Treatise on electricity and magnetism' has done so much to familiarize students of physics with quaternion notations, that it seems impossible that this subject should ever again be entirely divorced from the methods of multiple algebra. I wish that I could say as much of astronomy. It is, I think, to be regretted, that the oldest of the scientific applications of mathematics, the most dignified, the most conservative, should keep so far aloof from the youngest of mathematical methods; and standing, as I do to-day, by some chance, among astronomers, although not of the guild, I cannot but endeavor to improve the opportunity by expressing my conviction of the advantages which astronomers might gain by employing some of the methods of multiple algebra. A very few of the fundamental notions of a vector analysis, the addition of vectors and what quaternionists would call 'the scalar part and the vector part of the product of two vectors' (which may be defined without the definition of the quaternion), — these three notions, with some four fundamental properties relating to them, are sufficient to reduce enormously the labor of mastering such subjects as the elementary theory of orbits, the determination of an orbit from three observations, the differential equations which are used in determining the best orbit from an indefinite number of observations by the method of least squares, or those which give the perturbations when the elements are treated as variable. In all these subjects, the analytical work is greatly simplified, and it is far easier to get the best form for numerical calculation than in the use of the ordinary analysis."

Then followed illustrations of the various methods of applying multiple algebra to different classes of problems, and the paper closed as follows: "But I do not so much desire to call your attention to the diversity of the applications of multiple algebra, as to the simplicity and unity of its principles. The student of multiple algebra suddenly finds himself freed from various restrictions to which he has been accustomed. To many, doubtless, this liberty seems like an invitation to license. Here is a boundless field in which caprice may riot. It is not strange if some look with distrust for the result of such an experiment. But the further we advance, the more evident it becomes that this, too, is a realm subject to law. The more we study the subject, the more we find all that is most useful and beautiful attaching itself to a few central principles. We begin by studying 'multiple algebras;' we end, I think, by studying 'multiple algebra.'"

SEAT OF THE ELECTROMOTIVE FORCE.

PROFESSOR BRACKETT'S address was essentially a résumé of the history of the investigations to find the source of the current in galvanic batteries. No attempt was made to settle the question, which has been so long a bone of contention.

The address was so purely historical in its nature, and, withal, was so condensed and concise, that any abstract would be necessarily little more than an index of its contents. Those who are interested in the subject must await its publication in full in the 'Proceedings' of the association.

Galvani's two accidental discoveries were made in 1789: the one was the influence of an electrical machine in causing contractions in a frog's legs, and the other the production of sufficient electricity to cause the contraction by touching two joined strips of copper and zinc to the moist animal tissues. Naturally from these results there arose a theory of the identity of nerve-force and electricity, — the so-called animal variety of electricity. While this controversy, soon to subside, was started among physiologists, a much more

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